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Put option prices as joint distribution functions in strike and maturity : the Black-Scholes case

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Abstract : For a large class of \mathbb{R}_+ valued, continuous local martingales $(M_t, t \geq 0)$, with $M_0 = 1$ and $M_\infty = 0$, the put quantity : $\Pi_M(K, t) = E((K - M_t)^+)$ turns out to be the distribution function in both variables K and t , for $K \leq 1$ and $t \geq 0$, of a probability γ_M on $[0, 1] \times [0, \infty[$. In this paper, the first in a series of three, we discuss in detail the case where $M_t = \mathcal{E}_t := \exp(B_t - \frac{t}{2})$, for $(B_t, t \geq 0)$ a standard Brownian motion.

Keywords : First and last passage times, pseudo-inverse, local time-space calculus, Black-Scholes set up.

1 Introduction

1.1 Throughout this paper, we consider a generic continuous local martingale $(M_t, t \geq 0)$, taking values in \mathbb{R}_+ , and such that :

$$M_0 = 1, \lim_{t \rightarrow \infty} M_t = 0 \quad \text{a.s.} \tag{1.1}$$

To such an $(M_t, t \geq 0)$, we associate the function ¹ $\Pi_M : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by :

$$\Pi_M(K, t) := E((K - M_t)^+), \quad 0 \leq K \leq 1, t \geq 0 \quad (1.2)$$

Note that this function is separately increasing in K and t (since, $(K - x)^+$ being convex, $((K - M_t)^+, t \geq 0)$ is a submartingale). Furthermore, we have :

$$(i) \quad \begin{aligned} \Pi_M(K, 0) &= 0 \quad (0 \leq K \leq 1) \quad \text{and} \\ \Pi_M(K, \infty) &:= \lim_{t \rightarrow \infty} \Pi_M(K, t) = K \end{aligned} \quad (1.3)$$

(from the dominated convergence Theorem)

$$(ii) \quad \Pi_M(K, t) = K P(\mathcal{G}_K \leq t) \quad (1.4)$$

$$\text{with} \quad \mathcal{G}_K := \sup\{t \geq 0 ; M_t = K\} \quad (1.5)$$

(see [1], [5], [6], [7], [8]). In particular :

$$\Pi_M(1, t) = P(\mathcal{G}_1 \leq t) \quad (1.6)$$

Thus, $(\Pi_M(K, \infty), K \leq 1)$, resp. $(\Pi_M(1, t), t \geq 0)$ is a distribution function on $[0, 1]$, resp. on $[0, \infty[$; more precisely, these functions are, respectively, the distribution function of U , a standard uniform variable on $[0, 1]$, and of \mathcal{G}_1 .

To illustrate (1.6), let us recall that in the case $(M_t = \mathcal{E}_t := \exp(B_t - \frac{t}{2}), t \geq 0)$, with $(B_t ; t \geq 0)$ a standard Brownian motion it was shown in [5] that : $\mathcal{G}_1 \stackrel{(\text{law})}{=} 4B_1^2$; hence, (1.6) reads, in this case :

$$\Pi_{\mathcal{E}}(1, t) = P(4B_1^2 \leq t) \quad (1.7)$$

This formula may also be checked directly from the Black-Scholes formula.

1.2 It is thus a natural question to ask whether the function of K and t : $(\Pi_M(K, t) ; K \leq 1, t \geq 0)$ is the distribution function of a probability on $[0, 1] \times [0, \infty[$ which, assuming it exists, we shall denote by $\gamma(= \gamma_M)$. If so, we have :

$$E((K - M_t)^+) = \gamma([0, K] \times [0, t]), \quad K \leq 1, t \geq 0 \quad (1.8)$$

1.3 Here is our strategy to attack this question. Note that by Fubini :

$$E((K - M_t)^+) = \int_0^K P(M_t \leq x) dx \quad (1.9)$$

¹It is the "put" associated to the local martingale (M_t) , with strike K , and maturity t .

Assume that there exists, for every $x < 1$, a r.v. $Y_x \geq 0$ such that :

$$P(M_t \leq x) = P(Y_x \leq t) \quad (x < 1, t \geq 0) \quad (1.10)$$

We shall call this collection $(Y_x, x \leq 1)$ of r.v.'s, provided it exists, a pseudo-inverse of $(M_t, t \geq 0)$.

To justify this terminology, think of $(M_t, t \geq 0)$ as a process with a tendency to decrease, since it is a supermartingale which converges to 0 as $t \rightarrow \infty$. Now, replace $(M_t, t \geq 0)$ by a decreasing process $(D_t, t \geq 0)$; then, if $(\theta_d, d \geq 0)$ is the inverse of D , we have : $P(D_t \leq x) = P(\theta_x \leq t)$.

Let us go back to (1.9), and assume that $(M_t, t \geq 0)$ admits a pseudo-inverse $(Y_x, x \leq 1)$. Then, plugging (1.10) in (1.9), we find that γ exists, and it is the probability :

$$\gamma(dx, dt) = dx P(Y_x \in dt) \quad \text{on } [0, 1] \times [0, \infty[$$

Note that, a priori, we do not know the existence of $(Y_x, x \leq 1)$ as a process ; if such a process exists, then γ is the law of :

$$(U, Y_U) \quad (1.11)$$

where U is uniform on $[0, 1]$, independent of $(Y_x, x < 1)$.

1.4 In practice, most of the time, the function :

$$(K, t) \rightarrow \Pi_M(K, t)$$

is regular ; if so, we find that $(M_t, t \geq 0)$ admits a pseudo-inverse if and only if :

$$\frac{\partial^2}{\partial K \partial t} (\Pi_M(K, t)) \geq 0 \quad (1.12)$$

and then :

$$\gamma(dK, dt) = dK P(Y_K \in dt) = \left(\frac{\partial^2}{\partial K \partial t} (\Pi_M(K, t)) \right) dK dt \quad (1.13)$$

1.5 In Section 2, we shall develop this program in the case $M_t = \mathcal{E}_t := \exp(B_t - \frac{t}{2}), t \geq 0$ where $(B_t, t \geq 0)$ is a standard Brownian motion started at 0. In particular, we prove the existence of a pseudo-inverse for $(\mathcal{E}_t, t \geq 0)$.

1.6 In Part II, (in preparation) we aim to show the existence of a pseudo-inverse for a large number of Markovian martingales $(M_t, t \geq 0)$. In Part III, further classes of martingales shall be shown to admit pseudo-inverses.

2 The Black-Scholes paradigm

2.1 In this Section, $(B_t, t \geq 0)$ denotes a standard Brownian motion started at 0 and $(\mathcal{E}_t, t \geq 0)$ is the exponential martingale defined by :

$$\mathcal{E}_t := \exp\left(B_t - \frac{t}{2}\right), \quad t \geq 0 \quad (2.1)$$

Note that $\mathcal{E}_0 = 1$ and $\mathcal{E}_t \xrightarrow[t \rightarrow \infty]{} 0$ a.s. We define, for $0 \leq K \leq 1$ and $t \geq 0$:

$$\Pi_{\mathcal{E}}(K, t) := E((K - \mathcal{E}_t)^+) \quad (2.2)$$

Theorem 2.1 *There exists a probability, which we shall denote by γ , on $[0, 1] \times [0, \infty[$ such that :*

$$\Pi_{\mathcal{E}}(K, t) = \gamma([0, K] \times [0, t]) \quad (2.3)$$

(see point ii) of Proposition 2.7 for a description of the density of γ).

In order to prove Theorem 2.1 and to describe γ , we start with :

Lemma 2.2 *Denote by \mathcal{N} the distribution function of the standard Gaussian variable :*

$$\mathcal{N}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy \quad (x \in \mathbb{R}) \quad (2.4)$$

Then :

i) *To any $a, b > 0$, one can associate a r.v. $Y_{a,b}$, taking values in $[0, \infty[$, such that :*

$$P(Y_{a,b} \leq t) = \mathcal{N}\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right), \quad t \geq 0 \quad (2.5)$$

ii) *The density $f_{Y_{a,b}}$ of $Y_{a,b}$ is given by :*

$$f_{Y_{a,b}}(t) = \frac{1}{\sqrt{2\pi}} e^{ab} \cdot \left(\frac{a}{2\sqrt{t}} + \frac{b}{2t^{\frac{3}{2}}}\right) \exp\left(-\frac{1}{2}\left(a^2t + \frac{b^2}{t}\right)\right) \quad (2.6)$$

iii) *Let us define :*

$$T_b^{(a)} := \inf\{t \geq 0 ; B_t + at = b\} \quad (2.7)$$

$$G_b^{(a)} := \sup\{t \geq 0 ; B_t + at = b\} \quad (2.8)$$

Then :

$$P(Y_{a,b} \in dt) = \frac{1}{2} \{P(T_b^{(a)} \in dt) + P(G_b^{(a)} \in dt)\} \quad (2.9)$$

We note that this formula (2.9) allows to define the law of a process $(Y_{a,b} ; b \geq 0)$ obtained as a fair coin toss of $(T_b^{(a)} ; b \geq 0)$ on one hand and $(G_b^{(a)} ; b \geq 0)$ on the other hand.

Remark 2.3

i) It may be worth mentioning that this point *i)* of Lemma 2.2 admits a wide extension, since to any distribution function F on \mathbb{R} , and any $a, b > 0$, we can associate a new distribution function $F_{a,b}$ on \mathbb{R}_+ via :

$$F_{a,b}(t) = F\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right) \quad (t \geq 0)$$

However, the particular case $F = \mathcal{N}$ fits extremely well with our discussion.

ii) We note that $\frac{1}{Y_{a,b}} \stackrel{(\text{law})}{=} Y_{b,a}$, which may be deduced from either (2.5), (2.6) or (2.9).

Proof of Lemma 2.2

a. Points *i)* and *ii)* of Lemma 2.2 are immediate since :

- $\lim_{t \downarrow 0} \mathcal{N}\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right) = 0$
- $\lim_{t \uparrow \infty} \mathcal{N}\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right) = 1$
- $\frac{\partial}{\partial t} \mathcal{N}\left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}} \left(\frac{a}{2\sqrt{t}} + \frac{b}{2t^{\frac{3}{2}}}\right) \exp -\frac{1}{2} \left(a\sqrt{t} - \frac{b}{\sqrt{t}}\right)^2 \geq 0$
 $= f_{Y_{a,b}}(t)$

b. We now prove point *iii)* of Lemma 2.2 :

We shall use the well-known formulae (see e.g. [6])

$$P(T_b^{(a)} \in dt) = \frac{b dt}{\sqrt{2\pi t^3}} \exp -\frac{1}{2t}(b - at)^2 \quad (a, b > 0, t \geq 0) \quad (2.10)$$

$$P(G_b^{(a)} \in dt) = \frac{a dt}{\sqrt{2\pi t}} \exp -\frac{1}{2t}(b - at)^2 \quad (a, b > 0, t \geq 0) \quad (2.11)$$

(2.9) is now an immediate consequence of (2.10), (2.11) and (2.6).

c. Another proof of point *iii)* of Lemma 2.2 :

Let us denote, for $\nu > 0$, $(\mathcal{E}_t^{(2\nu)} := \exp(2\nu B_t - 2\nu^2 t), t \geq 0)$. It is proven in [6] Theorem 1, that, for $A \geq 1$:

$$P(T_{\log A}^{(\nu)} \leq t) = E(\mathcal{E}_t^{(2\nu)} 1_{\mathcal{E}_t^{(2\nu)} > A^{2\nu}}) + A^{2\nu} P(\mathcal{E}_t^{(2\nu)} > A^{2\nu})$$

and for $A \geq 0$:

$$P(G_{\log A}^{(\nu)} \leq t) = E(\mathcal{E}_t^{(2\nu)} 1_{\mathcal{E}_t^{(2\nu)} > A^{2\nu}}) - A^{2\nu} P(\mathcal{E}_t^{(2\nu)} > A^{2\nu})$$

Thus, by addition :

$$\begin{aligned}
\frac{1}{2}\{P(T_{\log A}^{(\nu)} \leq t) + P(G_{\log A}^{(\nu)} \leq t)\} &= E(\mathcal{E}_t^{(2\nu)} 1_{\mathcal{E}_t^{(2\nu)} > A^{2\nu}}) \\
&= P(e^{2\nu B_t + 2\nu^2 t} > A^{2\nu}) \quad (\text{from Cameron-Martin formula}) \\
&= P(B_t > \log A - \nu t) = P\left(B_1 > \frac{\log A - \nu t}{\sqrt{t}}\right) \\
&= 1 - \mathcal{N}\left(\frac{\log A}{\sqrt{t}} - \nu\sqrt{t}\right) = \mathcal{N}\left(\nu\sqrt{t} - \frac{\log A}{\sqrt{t}}\right) \tag{2.12}
\end{aligned}$$

(2.6) is now an immediate consequence of (2.12), with $b = \log A$ and $\nu = a$, by derivation with respect to t .

Proof of Theorem 2.1 From (1.8), we denote, for $K \leq 1$ and $t \geq 0$:

$$\begin{aligned}
\Pi_{\mathcal{E}}(K, t) &:= E((K - \mathcal{E}_t)^+) = \int_0^K P(\mathcal{E}_t \leq x) dx \\
&= \int_0^K \mathcal{N}\left(\frac{\log x}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) dx \\
(\text{since } P(\mathcal{E}_t \leq x) &= P(e^{B_t - \frac{t}{2}} \leq x) = P(B_1 < \frac{\log x}{\sqrt{t}} + \frac{\sqrt{t}}{2})) \\
&= \int_0^K P(Y_{\frac{1}{2}, \log \frac{1}{x}} \leq t) dx \quad (\text{from Lemma 2.2}) \tag{2.13}
\end{aligned}$$

Hence :

$$\frac{\partial^2}{\partial K \partial t} \Pi_{\mathcal{E}}(K, t) = \frac{\partial}{\partial t} P(Y_{\frac{1}{2}, \log \frac{1}{K}} \leq t) = f_{Y_{\frac{1}{2}, \log \frac{1}{K}}}(t) \geq 0$$

This ends the proof of Theorem 1.

2.2 Descriptions of the probability γ

2.2.1 First description of γ : conditioning with respect to U

Proposition 2.4 *The probability γ on $[0, 1] \times [0, \infty[$ is the law of the pair :*

$$(U, Y_{\frac{1}{2}, \log \frac{1}{U}}) \tag{2.14}$$

where U is uniform on $[0, 1]$, independent of the process $(Y_{\frac{1}{2}, b}, b > 0)$.

We now describe in words this probability viewed from (2.14) : it is the law of a two components r.v. ; the first component is the choice of a level out of the moneyness for a put, or the choice of a strike $K < 1$, uniformly on $[0, 1]$. Given this level, we construct the second variable on the outcome of a fair coin toss as either the first passage time of

the stock price under the share measure to the level $\left(\frac{1}{K}\right)$, or the last passage time of the stock price under the share measure to level $\left(\frac{1}{K}\right)$.

Proof of Proposition 2.4

We have, for $K \leq 1$ and $t \geq 0$:

$$\begin{aligned}
P(U \leq K, Y_{\frac{1}{2}, \log \frac{1}{U}} \leq t) &= \int_0^K du P(Y_{\frac{1}{2}, \log \frac{1}{u}} \leq t) \quad (\text{as explained just above}) \\
&= \int_0^K \mathcal{N}\left(\frac{\sqrt{t}}{2} + \frac{\log u}{\sqrt{t}}\right) du \\
&= \int_0^K P(\mathcal{E}_t \leq u) \quad (\text{since } \mathcal{E}_t \stackrel{(\text{law})}{=} \exp\left(\sqrt{t}B_1 - \frac{t}{2}\right), \text{ for fixed } t) \\
&= E[(K - \mathcal{E}_t)^+] \quad (\text{from (1.9)}) \\
&= \gamma([0, K] \times [0, t]) \quad (\text{from (2.3)})
\end{aligned}$$

The density of γ with respect to the Lebesgue measure on $[0, 1] \times \mathbb{R}_+$ given by (2.23) may also be obtained from the preceding relations.

2.2.2 Second description of γ : conditioning with respect to \mathcal{G}_1

Proposition 2.5 *The probability γ on $[0, 1] \times [0, \infty[$ is the law of the pair :*

$$(\exp(-2\epsilon) \vee \exp(-\sqrt{8\epsilon'B_1^2}), \quad 4B_1^2) \quad (2.15)$$

where B_1, ϵ, ϵ' are independent, with ϵ and ϵ' two standard exponential variables.

Remark 2.6 Upon comparing Propositions 2.4 and 2.5, it is quite natural to look for some understanding of the implied identities in laws between the first, resp. second, components of (2.15) and (2.14) ; precisely, we wish to check directly that :

$$(\exp(-2\epsilon)) \vee (\exp(-\sqrt{8\epsilon'B_1^2})) \stackrel{(\text{law})}{=} U \quad (2.16)$$

$$Y_{\frac{1}{2}, \log \frac{1}{U}} \stackrel{(\text{law})}{=} 4B_1^2 \quad (\stackrel{(\text{law})}{=} \mathcal{G}_0^{(-\frac{1}{2})}) \quad (2.17)$$

a. We now prove (2.16)

- First, we have :

$$\sqrt{2\epsilon'B_1^2} \stackrel{(\text{law})}{=} \epsilon \quad (2.18)$$

Indeed :

$$P(\sqrt{2\epsilon'B_1^2} > x) = P(\epsilon' > \frac{x^2}{2B_1^2}) = E(\exp(-\frac{x^2}{2B_1^2})) \stackrel{(a)}{=} E(\exp(-\frac{x^2}{2} T_1)) \stackrel{(b)}{=} \exp(-x)$$

since T_1 , the first hitting time of 1 by $(B_t, t \geq 0)$ is distributed as $\frac{1}{B_1^2}$, hence (a), and the Laplace transform of T_1 is well known to be given by (b).

- Since $\exp(-2\epsilon) \stackrel{(\text{law})}{=} U^2$, we have, from (2.18) :

$$(\exp(-2\epsilon)) \vee (\exp(-\sqrt{8\epsilon' B_1^2})) \stackrel{(\text{law})}{=} U^2 \vee (U')^2$$

with U and U' uniform on $[0, 1]$ and independent. But, for $y \in [0, 1]$:

$$P(U^2 \vee (U')^2 \leq y) = [P(U^2 \leq y)]^2 = (\sqrt{y})^2 = y$$

We have proven (2.16).

b. We now prove (2.17)

We have for every $t \geq 0$:

$$\begin{aligned} P(Y_{\frac{1}{2}, \log \frac{1}{U}} \leq t) &= \int_0^1 P(Y_{\frac{1}{2}, \log \frac{1}{u}} \leq t) du \quad (\text{after conditioning by } U = u) \\ &= \int_0^1 \mathcal{N}\left(\frac{\sqrt{t}}{2} + \frac{\log u}{\sqrt{t}}\right) du \quad (\text{from Lemma 2.2}) \\ &= E((1 - \mathcal{E}_t)^+) \quad (\text{from (2.13)}) \\ &= P(4B_1^2 \leq t) \quad (\text{from the classical Black-Scholes formula, see (1.7)}) \end{aligned}$$

We now prove Proposition 2.5

Conditioning on B_1^2 and using the explicit formula for the density of B_1^2 :

$$f_{B_1^2}(z) = \frac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}} 1_{z>0}$$

we have, for $K \leq 1$ and $t \geq 0$:

$$\begin{aligned} &P(\exp(-2\epsilon) \vee (\exp(-\sqrt{8\epsilon' B_1^2})) \leq K, 4B_1^2 \leq t) \\ &= \int_0^{\frac{t}{4}} \frac{dz}{\sqrt{2\pi z}} e^{-\frac{z}{2}} P((\exp(-2\epsilon)) \vee (\exp(-z\sqrt{8\epsilon'})) \leq K) \\ &= \int_0^{\frac{t}{4}} \frac{dz}{\sqrt{2\pi z}} e^{-\frac{z}{2}} P((\exp(-2\epsilon)) \leq K) P\left(\epsilon' > \frac{\log^2(\frac{1}{K})}{8z^2}\right) \\ &= \sqrt{K} \int_0^{\frac{t}{4}} \frac{dz}{\sqrt{2\pi z}} e^{-\frac{z}{2}} \exp\left(-\frac{(\log \frac{1}{K})^2}{8z^2}\right) \quad (\text{since } \exp(-\epsilon) \stackrel{(\text{law})}{=} U) \\ &= \sqrt{K} E\left(1_{B_1^2 \leq \frac{t}{4}} \cdot \exp - \frac{\log^2(\frac{1}{K})}{8B_1^2}\right) \\ &= E[(K - \mathcal{E}_t)^+] \end{aligned} \tag{2.19}$$

where the last equality follows from Theorem 2 of [6] which asserts that for $K \leq 1$:

$$E[(K - \mathcal{E}_t)^+] = \sqrt{K} E \left\{ 1_{4B_1^2 \leq t} \exp \left(-\frac{(\log(\frac{1}{K}))^2}{8B_1^2} \right) \right\} \quad (2.20)$$

■

Another proof of Proposition 2.5

We have, from (2.11) with $a = 1/2$ and $K < 1$:

$$P(\mathcal{G}_K \in ds) = \frac{\frac{1}{2}}{\sqrt{2\pi s}} \exp -\frac{1}{2s} \left(\log K + \frac{s}{2} \right)^2 ds$$

Hence :

$$\begin{aligned} KP(\mathcal{G}_K \in ds) &= \sqrt{K} \left(\exp \left(-\frac{(\log \frac{1}{K})^2}{2s} \right) \right) P(\mathcal{G}_1 \in ds) \\ &= \exp \left(\frac{1}{2} \log K \right) \exp \left(-\frac{(\log(\frac{1}{K}))^2}{2s} \right) P(\mathcal{G}_1 \in ds) \\ &= P \left(\mathfrak{e} > -\frac{1}{2} \log K \right) P \left(\mathfrak{e}' > \frac{(\log \frac{1}{K})^2}{2s} \right) P(\mathcal{G}_1 \in ds) \\ &= P \left(\exp(-2\mathfrak{e}) \vee \exp(-\sqrt{2s} \mathfrak{e}') < K \right) P(\mathcal{G}_1 \in ds) \\ &= P \left(\exp(-2\mathfrak{e}) \vee \exp(-\sqrt{8B_1^2} \mathfrak{e}') < K \right) P(4B_1^2 \in ds) \end{aligned}$$

since $\mathcal{G}_1 \stackrel{(\text{law})}{=} 4B_1^2$ (see [5]).

Hence, since, from (1.4), $E((K - \mathcal{E}_s)^+) = K P(\mathcal{G}_K \leq s)$, one has :

$$\begin{aligned} \gamma(dK, ds) &= \frac{\partial^2}{\partial K \partial s} (K P(\mathcal{G}_K \leq s)) dK ds \\ &= P \left(\exp(-2\mathfrak{e}) \vee \exp(-\sqrt{8B_1^2} \mathfrak{e}') \in dK \right) P(4B_1^2 \in ds) \end{aligned}$$

which is Proposition 2.5

2.2.3 Third description of γ : its relation with local time-space calculus

Let us define $(L_s^K ; K \geq 0, s \geq 0)$ the jointly continuous family of local times of the martingale $(\mathcal{E}_s, s \geq 0)$. This family is characterized by the occupation density formula :

$$\int_0^t f(\mathcal{E}_s) d\langle \mathcal{E} \rangle_s = \int_0^\infty f(K) L_t^K dK$$

for every Borel and positive function f . Here $(\langle \mathcal{E} \rangle_s, s \geq 0)$ denotes the bracket i.e. the quadratic variation process of $(\mathcal{E}_s, s \geq 0)$ and we have :

$$d \langle \mathcal{E} \rangle_s = \mathcal{E}_s^2 ds$$

The Itô-Tanaka formula yields, for $K \leq 1$:

$$E((K - \mathcal{E}_t)^+) = \frac{1}{2} E(L_t^K) \quad (2.21)$$

As a consequence of (2.21), we obtain point *i*) of the following :

Proposition 2.7

i) The probability γ on $[0, 1] \times [0, \infty[$ admits a density f_γ which satisfies :

$$\gamma(dK, dt) = \frac{1}{2} \left(\frac{\partial^2}{\partial K \partial t} E(L_t^K) \right) dK dt \quad (0 \leq K \leq 1, t \geq 0) \quad (2.22)$$

ii) A closed form of f_γ is :

$$f_\gamma(K, t) = \frac{1}{2\sqrt{2\pi Kt}} \left(\frac{1}{2} - \frac{\log K}{t} \right) \exp \left(-\frac{(\log K)^2}{2t} - \frac{t}{8} \right) 1_{[0,1]}(K) \cdot 1_{[0,\infty[}(t) \quad (2.23)$$

Proof of point *ii*) of Proposition 2.7

In [6], we obtained the following explicit formula for $E(L_t^K)$:

$$E(L_t^K) = \frac{\sqrt{K}}{\sqrt{2\pi}} \int_0^t \frac{ds}{\sqrt{s}} \exp \left(-\frac{(\log(K))^2}{2s} - \frac{s}{8} \right) \quad (2.24)$$

Hence :

$$\frac{\partial^2}{\partial K \partial t} E(L_t^K) = \frac{1}{\sqrt{2\pi t}} \frac{\partial}{\partial K} \left(\sqrt{K} \exp \left(-\frac{(\log K)^2}{2t} - \frac{t}{8} \right) \right) \quad (2.25)$$

and (2.23) is an easy consequence of (2.25) and (2.22).

2.2.4 Relation with a result by N. Eisenbaum (see [2a] and [2b]).

We now relate the above description of γ as in Proposition 2.7 with the definition-formula established in [2b] :

$$\int_{-\infty}^{\infty} \int_0^t f(K, s) d_{K,s}(L_s^K) = \int_0^t f(\mathcal{E}_s, s) d\mathcal{E}_s + \int_{1-t}^1 f(\mathcal{E}_{1-s}, 1-s) d_s \mathcal{E}_{1-s} \quad (t \leq 1) \quad (2.26)$$

This formula which is the translation for $X_s = \mathcal{E}_s$ of the formula found in Theorem 2.2 of [2b] for a general reversible semi-martingale. Here, on the RHS of (2.26), the second stochastic integral is taken with respect to the natural filtration of $\hat{\mathcal{E}}_s = \mathcal{E}_{1-s}$, which is, of

course, that of $\widehat{B}_s = B_{1-s}$. We take f bounded, Borel, with support in $[0, 1]_K \times [0, 1]_s$. In order to relate formula (2.26) with Proposition 2.7, we note that :

$$\text{a)} \int_0^1 \int_0^1 f(K, s) \gamma(dK, ds) = \frac{1}{2} E \left(\int_0^1 \int_0^1 f(K, s) d_{K,s}(L_s^K) \right)$$

which follows from (2.21), and the monotone class Theorem.

b) From formula (2.26) and the fact that $(\mathcal{E}_s, t \geq 0)$ is a martingale, we deduce :

$$E \left(\int_0^1 \int_0^1 f(K, s) d_{K,s}(L_s^K) \right) = E \left(\int_0^1 f(\widehat{\mathcal{E}}_s, 1-s) d\widehat{\mathcal{E}}_s \right) \quad (2.27)$$

which we shall compute explicitly thanks to the semi martingale decomposition of $(\widehat{\mathcal{E}}_s, s \leq 1)$ in its own filtration. This is presented in the next :

Proposition 2.8

i) The canonical decomposition of $(B_t, t \leq 1)$ in the filtration

$\mathcal{B}_t^{(1)} := \sigma(B_s, s \leq t) \vee \sigma(B_1)$ is :

$$B_t = B_t^* + \int_0^t \frac{B_1 - B_s}{1-s} ds \quad (2.28)$$

where $(B_t^*, t \leq 1)$ is a $(\mathcal{B}_t^{(1)}, t \leq 1)$ Brownian motion.

ii) The canonical decomposition of $\widehat{B}_t = B_{1-t}$ in its own filtration is :

$$\widehat{B}_t = B_{1-t} = B_1 + \beta_t^* - \int_0^t \frac{B_{1-s}}{1-s} ds \quad (2.29)$$

where $(\beta_t^*, t \leq 1)$ is a Brownian motion in $\{\widehat{\mathcal{B}}_t := \sigma(\widehat{B}_u, u \leq t), t \leq 1\}$.

iii) The canonical decomposition of $\widehat{\mathcal{E}}_t$ in $\widehat{\mathcal{B}}_t$ is :

$$d\widehat{\mathcal{E}}_t = \widehat{\mathcal{E}}_t \left\{ d\beta_t^* + dt \left(1 - \frac{B_{1-t}}{1-t} \right) \right\} \quad (2.30)$$

Proof of Proposition 2.8

i) is well-known, see, e.g., Jeulin-Yor ([4], 1979) or Itô (1976) : Extension of Stochastic Integrals ([3])

ii) may be deduced from i), when i) is applied to $\beta_t = B_{1-t} - B_1$. Actually, formula (2.29) appears in [2a], bottom of p. 308

iii) follows from ii), thanks to Itô's formula.

Comments **a)** We are grateful to N. Eisenbaum (personal communication) for pointing out formula (2.29), which allowed to correct our original wrong derivation of the canonical decomposition of $\widehat{B}_t = B_{1-t}$:

$$\widehat{B}_t = B_{1-t} = B_1 + \widetilde{\beta}_t + \int_0^t \frac{du}{u} (\widehat{B}_u - B_1) \quad (2.31)$$

Indeed, by time-reversal from (2.28) in Proposition 2.8, there is the identity :

$$\widehat{B}_t = B_{1-t} = B_1 + \widetilde{\beta}_t + \int_0^t \frac{du}{u} (\widehat{B}_u - B_1)$$

where $\widetilde{\beta}_t = B_{1-t}^* - B_1^*$ is a Brownian motion, but is not the canonical decomposition of \widehat{B}_t in $\widehat{\mathcal{B}}_t$ (for a discussion of non-canonical decomposition of Brownian motion, see, e.g. Yor [9], Chiu [10], Hibino, Hitsuda, Muraoka [11]).

b) A slightly different derivation of (2.29) consists in remarking that $B_{1-t} = (1-t)B_1 + b(t)$ with $(b(t), 0 \leq t \leq 1)$ a standard Brownian bridge.

From *i)*, this Brownian bridge admits the decomposition :

$$b(t) = \beta_t^* - \int_0^t \frac{b(s)}{1-s} ds$$

Thus we obtain :

$$\begin{aligned} B_{1-t} &= (1-t)B_1 + \beta_t^* - \int_0^t \frac{ds}{1-s} (b(s) + (1-s)B_1 - (1-s)B_1) \\ &= (1-t)B_1 + \beta_t^* - \int_0^t \frac{B_{1-s}}{1-s} ds + tB_1 \\ &= B_1 + \beta_t^* - \int_0^t \frac{B_{1-s}}{1-s} ds \end{aligned}$$

which is precisely (2.29).

Now, plugging (2.30) in (2.27), and with the help of *i)*, we get :

$$\begin{aligned} \int_0^1 \int_0^1 f(K, s) \gamma(dK, ds) &= \frac{1}{2} E \left(\int_0^1 f(\mathcal{E}_s, s) \mathcal{E}_s \left(1 - \frac{B_s}{s} \right) ds \right) \\ &= \frac{1}{2} E \left(\int_0^1 \mathcal{E}_s f(\mathcal{E}_s, s) \left(\frac{1}{2} - \frac{\log \mathcal{E}_s}{s} \right) ds \right) \end{aligned}$$

which matches perfectly with our previous formula (2.23). Thus, we have established a close link with local time-space calculus as developed in ([2a] - [2b]).

2.3 An extension of Theorem 2.1

In the next statement, we shall replace the standard Brownian martingale $(\mathcal{E}_t, t \geq 0)$ by the semimartingale $(\mathcal{E}_t^{\sigma, -\nu} := \exp(\sigma B_t - \nu t), t \geq 0)$ ($\sigma \neq 0, \nu > 0$). Then we can show :

Theorem 2.9 *i) There exists a probability on $[0, 1] \times [0, \infty[$, which we shall denote by $\gamma_{\sigma, \nu}$ such that :*

$$\Pi_{\sigma, \nu}(K, t) := E((K - \mathcal{E}_t^{\sigma, -\nu})^+) = \gamma_{\sigma, \nu}([0, K] \times [0, t]) \quad (2.32)$$

ii) Moreover, $\gamma_{\sigma, \nu}$ is the law of :

$$(U, Y_{\frac{\nu}{|\sigma|}, \frac{1}{|\sigma|} \log \frac{1}{U}}) \quad (2.33)$$

where U is uniform on $[0, 1]$, independent of $(Y_{a,b}, a, b > 0)$ as introduced in Lemma 2.2.

Proof of Theorem 2.9

We may choose $\sigma > 0$, since $\sigma B_t \stackrel{(\text{law})}{=} -\sigma B_t$. Then, from the definition (2.26), we write, for $K \leq 1$, by Fubini :

$$\begin{aligned} E((K - \mathcal{E}_t^{\sigma, -\nu})^+) &= \int_0^K P(\mathcal{E}_t^{\sigma, -\nu} \leq x) dx \\ &= \int_0^K P\left(B_1 < \frac{\nu\sqrt{t}}{\sigma} + \frac{\log x}{\sigma\sqrt{t}}\right) dx \\ &= \int_0^K \mathcal{N}\left(\frac{\nu\sqrt{t}}{\sigma} + \frac{\log x}{\sigma\sqrt{t}}\right) dx \\ &= \int_0^K P\left(Y_{\frac{\nu}{\sigma}, \frac{1}{\sigma} \log \frac{1}{x}} \leq t\right) dx \quad (\text{from Lemma 2.2}) \end{aligned}$$

which implies points *i)* and *ii)* of Theorem 2.9. ■

Note that (2.33) corresponds to the first description in Section 2.2 of the particular case $\sigma = 1, \nu = \frac{1}{2}$. We would like to see whether there is a second description of $\gamma_{\sigma, \nu}$; in particular, what is the law of $Y_{\frac{\nu}{\sigma}, \frac{1}{\sigma} \log \frac{1}{U}}$? Let us denote by $f_{\sigma, \nu}$ the density of $Y_{\frac{\nu}{\sigma}, \frac{1}{\sigma} \log \frac{1}{U}}$. Then, we have :

Proposition 2.10 *The following formulae hold :*

$$i) \quad f_{\sigma, \nu}(t) = \frac{\sigma}{2\sqrt{2\pi t}} e^{-\frac{\nu^2 t}{2\sigma^2}} \left(1 + \left(2\frac{\nu}{\sigma^2} - 1\right) \int_0^\infty e^{-\mu x - \frac{x^2}{2\sigma^2 t}} dx\right) \quad (2.34)$$

where $\mu = 1 - \frac{\nu}{\sigma^2}$

ii) In particular, if $\frac{2\nu}{\sigma^2} = 1$, (this condition ensures that $(\mathcal{E}_t^{\sigma, -\nu}, t \geq 0)$ is a martingale !), we have :

$$f_{\sigma, \frac{\sigma^2}{2}}(t) = \frac{\sigma}{2\sqrt{2\pi t}} e^{-\frac{\sigma^2 t}{8}} \quad (2.35)$$

$\left(= f_{4B_1^2}(t) \text{ if } \sigma = 1 \text{ and } \nu = \frac{1}{2}, \text{ in agreement with Proposition 2.5, formula (2.17)} \right).$

Proof of (2.34)

From (2.12) and (2.33) we have :

$$f_{\sigma,\nu}(t)dt = \frac{1}{2} \int_0^1 du \left\{ P \left(T_{\frac{1}{\sigma} \log \frac{1}{u}}^{(\frac{\nu}{\sigma})} \in dt \right) + P \left(G_{\frac{1}{\sigma} \log \frac{1}{u}}^{(\frac{\nu}{\sigma})} \in dt \right) \right\}$$

Making the change of variable $\log \frac{1}{u} = x$ and using (2.10) and (2.11), we obtain :

$$\begin{aligned} f_{\sigma,\nu}(t) &= \frac{1}{2} \int_0^\infty e^{-x} \left\{ \left(\frac{x}{\sigma\sqrt{2\pi t^3}} + \frac{\nu}{\sigma\sqrt{2\pi t}} \right) e^{-\frac{(x-\nu t)^2}{2\sigma^2 t}} \right\} dx \\ &= \frac{\sigma}{2\sqrt{2\pi t}} \exp \left(-\frac{\nu^2 t}{2\sigma^2} \right) \int_0^\infty \exp \left(-x - \frac{x^2}{2\sigma^2 t} + \frac{x\nu}{\sigma^2} \right) \left(\frac{x}{t\sigma^2} + \frac{\nu}{\sigma^2} \right) dx \end{aligned}$$

We now introduce the parameter $\mu = 1 - \frac{\nu}{\sigma^2}$ and we compute the integral :

$$\begin{aligned} I &:= \int_0^\infty e^{-\mu x - \frac{x^2}{2\sigma^2 t}} \left(\frac{x}{t\sigma^2} + \frac{\nu}{\sigma^2} \right) dx \\ &:= I_1 + \frac{\nu}{\sigma^2} I_2 \quad \text{with} \\ I_1 &= \int_0^\infty e^{-\mu x - \frac{x^2}{2\sigma^2 t}} \frac{x}{t\sigma^2} dx = 1 - \mu \int_0^\infty e^{-\mu x - \frac{x^2}{2\sigma^2 t}} dx \end{aligned}$$

(after integrating by parts). Finally :

$$\begin{aligned} I &= I_1 + \frac{\nu}{\sigma^2} I_2 = 1 - \mu I_2 + \frac{\nu}{\sigma^2} I_2 \\ &= 1 + \left(\frac{2\nu}{\sigma^2} - 1 \right) \int_0^\infty e^{-\mu x - \frac{x^2}{2\sigma^2 t}} dx \quad \text{and :} \\ f_{\sigma,\nu}(t) &= \frac{\sigma}{2\sqrt{2\pi t}} \exp \left(-\frac{\nu^2 t}{2\sigma^2} \right) \left\{ 1 + \left(\frac{2\nu}{\sigma^2} - 1 \right) \int_0^\infty e^{-\mu x - \frac{x^2}{2\sigma^2 t}} dx \right\} \end{aligned}$$

This proves (2.33).

2.4 γ as a signed measure on $\mathbb{R}_+ \times \mathbb{R}_+$

In this paragraph, we extend the definition of γ to $[0, \infty[\times [0, \infty[$.

Proposition 2.11

i) There exists a signed measure γ on $[0, \infty[\times [0, \infty[$ such that :

$$\Pi_{\mathcal{E}}(K, t) := E((K - \mathcal{E}_t)^+) = \gamma([0, K] \times [0, t])$$

holds for all values $K, t \geq 0$. (2.36)

ii) γ admits on $\mathbb{R}_+ \times \mathbb{R}_+$ the density f_γ given by :

$$\begin{aligned} f_\gamma(K, t) &= \frac{1}{2\sqrt{2\pi Kt}} \left(\frac{1}{2} - \frac{\log K}{t} \right) \exp \left(-\frac{(\log K)^2}{2t} - \frac{t}{8} \right) 1_{[0, \infty[\times [0, \infty[}(K, t) \\ &= \frac{1}{2\sqrt{2\pi t}} \left(\frac{1}{2} + \frac{\log \frac{1}{K}}{t} \right) \exp \left(-\frac{1}{2t} \left(\log \frac{1}{K} - \frac{t}{2} \right)^2 \right) 1_{[0, \infty[\times [0, \infty[}(K, t) \end{aligned} \quad (2.37)$$

iii) Consequently, if $\gamma = \gamma^+ - \gamma^-$ is the decomposition of γ into its positive and negative parts, we have :

$$\gamma^+(dK, dt) = 1_{(K \leq e^{\frac{t}{2}})} \gamma(dK, dt) \quad (2.38)$$

In particular :

$$\gamma_{|(K, t; K \leq 1)}^+ = \gamma_{|(K, t; K \leq 1)}$$

is a probability.

iv) Formula (2.31) may be synthetized as follows :

- if $K < 1$:

$$\gamma(dK, dt) = \frac{dK}{2} \left\{ P(G_{\log(\frac{1}{K})}^{(\frac{1}{2})} \in dt) + P(T_{\log(\frac{1}{K})}^{(\frac{1}{2})} \in dt) \right\} \quad (2.39)$$

- if $K > 1$:

$$\gamma(dK, dt) = \frac{dK}{2} \left\{ P(G_{\log(\frac{1}{K})}^{(\frac{1}{2})} \in dt) - P(T_{\log(\frac{1}{K})}^{(\frac{1}{2})} \in dt) \right\} \quad (2.40)$$

Proof of Proposition 2.11

Points i) and ii) are easy to prove. Indeed, for every $K \geq 0$:

$$E((K - \mathcal{E}_t)^+) = \int_0^K P(\mathcal{E}_t < x) dx = \int_0^K \mathcal{N} \left(\frac{\sqrt{t}}{2} + \frac{\log x}{\sqrt{t}} \right) dx$$

This last formula implies the existence of γ and the density f_γ of γ is given by :

$$\begin{aligned} f_\gamma(K, t) &= \frac{\partial^2}{\partial K \partial t} \int_0^K \mathcal{N} \left(\frac{\sqrt{t}}{2} + \frac{\log x}{\sqrt{t}} \right) dx \\ &= \frac{\partial}{\partial t} \mathcal{N} \left(\frac{\sqrt{t}}{2} + \frac{\log K}{\sqrt{t}} \right) \\ &= \frac{1}{2\sqrt{2\pi Kt}} \left(\frac{1}{2} - \frac{\log K}{t} \right) \exp \left(-\frac{(\log K)^2}{2t} - \frac{t}{8} \right) \end{aligned}$$

Point *iii*) follows immediately.

Formulae (2.39) and (2.40) are obtained as in (2.12). Moreover, this time, we also need to use formulae from Theorem 1 of [6].

- If $A > 1$, then :

$$P(T_{\log A}^{(\nu)} \leq t) = E(\mathcal{E}_t^{(2\nu)} 1_{\mathcal{E}_t^{(2\nu)} > A^{2\nu}}) + A^{2\nu} P(\mathcal{E}_t^{(2\nu)} > A^{2\nu})$$

- If $A < 1$:

$$P(T_{\log A}^{(\nu)} \leq t) = E(\mathcal{E}_t^{(2\nu)} 1_{\mathcal{E}_t^{(2\nu)} < A^{2\nu}}) + A^{2\nu} P(\mathcal{E}_t^{(2\nu)} < A^{2\nu})$$

(see point *iii*) in Theorem 1 of [6]). ■

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